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Stability of Couette flow in nematic liquid crystals

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We study the stability of the cylindrical Couette flow in nematics when the director is parallel to the rotation axis. The contribution of the inertial coupling of velocity fluctuations (responsible for the Taylor instability in isotropic liquids) is shown to be destabilizing when the inner cylinder rotates faster than the outer one. However, the instability remains driven by the mechanisms first discovered by Pieranski & Guyon for the plane shear case and quite specific to nematics. This mechanism couples the different orientation fluctuations via viscous torques and the corresponding threshold is given by

$$s\tau_0 \sim 1,$$

where τ_0 is the time constant for the diffusion of orientation fluctuations. The contribution of inertia terms is measured by $2\omega_m \tau_v$, where τ_v is the time constant for the diffusion of velocity fluctuations. In usual nematics one has $\tau_v/\tau_0 \sim 10^{-5}$ so that corrections due to rotation are small in general. At different stages of the discussion differences between the case of nematics and that of isotropic liquids are pointed out. We also study the possibility of an oscillatory instability when α_3 is positive and large, where no stationary instability can occur.

1. Introduction

Nematic liquid crystals are ordered fluids made of elongated molecules aligned along a preferential direction. This mean orientation is a novel degree of freedom which adds to the usual set of hydrodynamic variables used to describe isotropic liquids. The continuous theory of nematics has been worked out by Frank (1958), Ericksen (1960), Leslie (1968), Parodi (1970); for a review see de Gennes (1974) or Stephen & Straley (1974); equations have been summarized at the beginning of appendix A. The coupling between orientation, labelled by a unit vector **n** called the 'director', and the flow has a very complicated structure; only certain simple situations are tractable which derive from symmetry considerations.

In a shear flow one can distinguish three fundamental positions for the director (Miezowicz 1946; figure 1). (*a*) In geometries (1) and (2), the director lies parallel to the plane of the shear and, quite intuitively, one understands that the flow exerts a viscous torque on the molecules. This torque tends to make the director rotate and when α_3 is negative an equilibrium position is found with **n** nearly parallel to the flow lines (Leslie 1968). (*b*) When the director is perpendicular to the shear plane [geometry (3)] molecules feel no viscous torque and the nematic behaves much like an isotropic liquid.

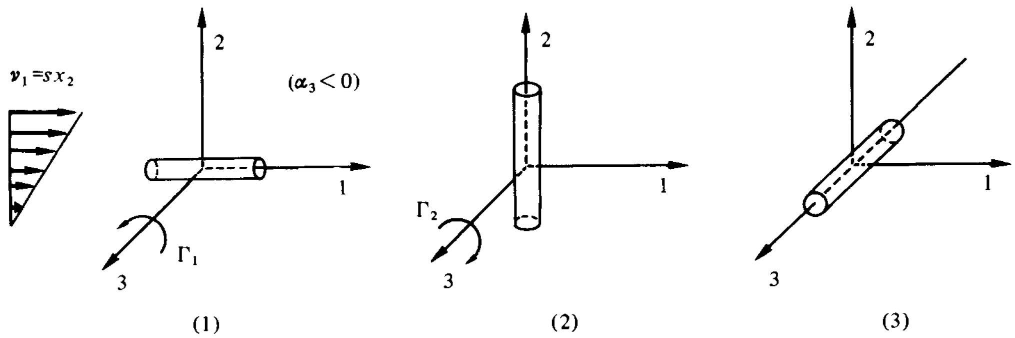

FIGURE 1. Viscous torques exerted on the director in the three Miezowicz geometries: (1) $\Gamma_1 = -\alpha_3 s$ ($\alpha_3 \gtrless 0$); (2) $\Gamma_2 = \alpha_2 s$ ($\alpha_2 < 0$); (3) $\Gamma_3 = 0$.

Since case (*b*) seemed less specific to nematics, it has received much less attention than case (*a*). As to the cylindrical Couette flow which is of interest here, solutions of the hydrodynamic equations have been searched for by Atkin & Leslie (1970) and more extensively by Currie (1970). In particular, Currie has pointed out that the flow is completely azimuthal if and only if **n** is either perpendicular or parallel to the rotation axis, which precisely corresponds to the distinction we have made above. Atkin & Leslie, as well as Currie, were only concerned with the search for solutions with **n** perpendicular to the rotation axis and did not tackle the problem of the stability of the flow.

First studies of the stability of the Couette flow dealt with incomplete hydrodynamic theories before the 1968 Leslie formulation; references may be found in a paper by Ericksen (1966). To our knowledge, no detailed theoretical approach has been worked out up to now. Several situations are possible depending on boundary conditions for the director and, more fundamentally, on the sign of the viscosity coefficient α_3. Pieranksi & Guyon (1975) have examined the case α_3 negative when the nematic is tangential to the cylinder surfaces ('planar' configuration). They have pointed out the mechanism responsible for an instability to the Taylor type which couples inertial forces to viscous forces specific to nematics. Cladis & Torza (1975, 1976) have performed experiments in homeotropic configuration (molecules perpendicular to the cylinder surfaces) using a nematic with α_3 positive. They have discovered several instabilities; however their interpretation has been questioned (Pieranski & Guyon 1976); indeed difficulties arise from the fact that, when α_3 is positive, the director has no longer an equilibrium position in the shear plane (Pikin 1973; de Gennes 1972, 1974; see also Currie 1970); moreover molecules can get out of this plane (Pieranski & Guyon 1974*a*) when the shearing rate exceeds a certain critical value (Pieranski, Guyon & Pikin 1976).

Case (*b*) may seem much more trivial and one could have expected a simple transposition of results concerning isotropic fluids. In fact, this is not the case since the direction perpendicular to the shear plane is quite privileged and since any fluctuation away from this direction reveals the anisotropic properties specific to nematics and, in particular, the existence of viscous torques. Indeed, while the planar Couette flow (simple shear flow) is linearly stable for isotropic liquids (see for example Landau & Lifshitz 1959) this is no longer the case for nematics. The instability which takes place follows from a constructive coupling between orientation fluctuations via the viscous torques (Pieranski & Guyon 1973, 1974*b*). A thorough theoretical approach has been developed recently (Leslie 1976; Manneville & Dubois-Violette 1976) which turns out to be in excellent agreement with experiments (Dubois-Violette *et al.* 1977); for a review see Manneville (1977).

Up to now, experiments in cylindrical geometry have been performed as an extension of planar shear flow experiments. They have been interpreted using results for the planar case, but as will be seen later this was quite legitimate since the effect of rotation was negligible. In this paper, we shall concentrate our attention on the modifications due to rotation, and moreover we shall restrict ourselves to the effect of rotation on instabilities specific to nematics. After a brief recall of the situation as it presents itself in the planar shear case, and in particular a rapid discussion of the instability mechanisms (§ 2), we shall write down the linearized hydrodynamic equations and discuss some simplifications (§ 3). The inertial coupling between velocity fluctuations is best visualized in the case of isotropic liquids; it will be discussed in § 4 together with a simplified model for the Taylor instability. This will help to explain the main difference with the case of nematics (§ 5) where orientation fluctuations play a dominant role. Finally, we shall make some predictions about the threshold value. This first qualitative approach will be completed by a more quantitative 'approximate normal mode analysis' which rests on effective torque equations and simplified analytical forms for the fluctuation profiles (§ 6). Order-of-magnitude estimates will show that, unless very high rotation rates are achieved, the effects of rotation are rather weak on instabilities specific to nematics; nevertheless, possible application will be briefly discussed.

For material with viscosity coefficient α_3 positive and large, it is known that no stationary instability can take place. In that case, we shall show in § 7 that oscillatory instability can occur.

2. Shear flow instabilities in nematics

When $\mathbf{n}$ is perpendicular to the shear plane [geometry (3) of figure 1] the nematic looks like an isotropic liquid. However, the study of the flow stability does not reduce to that for isotropic liquids. Indeed any fluctuation away from the direction of the unperturbed orientation feels a part of the viscous torques exerted in positions (1) and (2), proportional to its amplitude. These torques tend to make the orientation rotate as indicated in figures 2(*a*, *b*). Now as discovered by Pieranski & Guyon (1973), an instability may follow from the coupling between orientation fluctuation components through these viscous torques (figure 2*c*). Indeed, assume a fluctuation $n_2 > 0$. It induces a torque $\Gamma_2 > 0$ which tends to create a fluctuation $n_1 > 0$. Now this fluctuation induces a torque Γ_1 which reacts on n_2. The coupling turns out to be destabilizing when α_3 is negative and stabilizing in the opposite case. When α_3 is negative, the instability takes place only when the destabilizing mechanism is strong enough to overcome the stabilizing effect of the Frank orientational elasticity. Let us define the characteristic evolution time of orientation fluctuations:

$$\tau_0 = \gamma/Kq^2, \tag{2.1}$$

where γ is the orientational viscosity, K a typical Frank modulus and q the wave vector of the fluctuation.

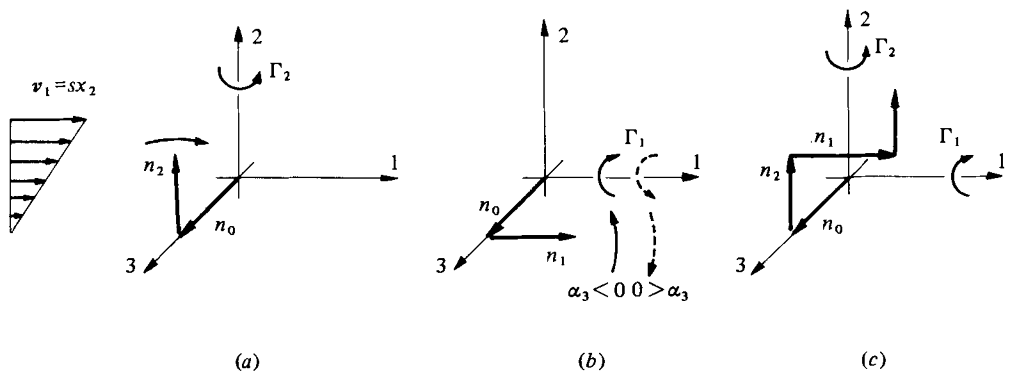

FIGURE 2. Pieranski–Guyon instability mechanism: owing to the fluctuation n_2, the shear flow exerts a torque Γ_2 on the director (a). This induces a fluctuation n_1 as indicated on (c). Owing to this fluctuation, a viscous torque Γ_1 appears, the sign of which depends on that of viscosity coefficient α_3 (b). For $\alpha_3 < 0$, this tends to increase the initial fluctuation n_2 (c).

From a purely dimensional point of view, one can parametrize the flow by the Ericksen number $Er = s\tau_0$ where τ_0 is evaluated for $q \sim \pi/d$, d being the cell thickness, and one can infer that the instability threshold s_c will be given by

$$s_c \tau_0 \sim 1 \quad \text{(where in our case } \tau_0 \text{ is typically of order 1 s).} \tag{2.2}$$

The basic mechanism just described leads to a distortion which is uniform in the plane of the flow (homogeneous instability: Pieranski & Guyon 1973). However a second instability mode is possible with a distortion periodic in the direction of the unperturbed orientation. It is associated with a secondary flow which takes the form of rolls parallel to the flow direction (roll instability: Pieranski & Guyon 1974a). In order to understand this kind of instability one has to take into account the contribution of velocity fluctuations to the viscous torques exerted on the molecules. In fact, spontaneous velocity fluctuations are much more rapid than orientation fluctuations. Indeed the characteristic time of velocity fluctuations is

$$\tau_v = \rho/\eta q^2, \tag{2.3}$$

where ρ is the density ($\sim 1\,\text{g/cm}^3$) and η the viscosity.

If one compares this with the orientation fluctuation time τ_0, one gets

$$\tau_v/\tau_0 = \rho K/\gamma\eta \sim 10^{-5}, \tag{2.4}$$

for typical values: $\gamma \sim \eta \sim 0{\cdot}1$ to 1 and $K \sim 10^{-6}$ cgs.

The evaluation could suggest that velocity fluctuations do not contribute to the instability mechanisms since they are not coherent over a sufficiently long time to be coupled with orientation. However, owing to the special form of the Leslie viscous stress tensor, in a shear flow, a non-uniform orientation induces *a viscous force* $\mathbf{F}^v$ *specific to nematics*. The motion equation for velocity fluctuations may then be simplified as

$$\rho \frac{d\mathbf{v}}{dt} = \eta\Delta\mathbf{v} + \mathbf{F}^v,$$

where $\mathbf{F}^v$ is very slowly varying (rate $\tau_0^{-1} \ll \tau_v^{-1}$). Then one may consider that the viscous force $\mathbf{F}^v$ induces the slowly varying secondary flow $\mathbf{v}$ roughly given by

$$\eta\Delta\mathbf{v} + \mathbf{F}^v \simeq 0.$$

Now this flow is non-uniform and contributes to torques exerted on the molecules. This contribution appears as intercalated in the basic sequence of fluctuation amplification.

$$\text{Orientation fluctuation} \rightarrow \underbrace{(\text{viscous force} \rightarrow \text{velocity fluctuation} \rightarrow)}_{\text{secondary flow effect}}$$

$$\rightarrow \text{viscous torque} \rightarrow \text{orientation fluctuation.}$$

This implies the notion of effective torques taking into account the effect of secondary flows induced by an orientation fluctuation via the viscous forces; accordingly, one can define effective viscosity coefficients α_2^* and α_3^*. Then, the existence of a roll instability specific to nematics will be subject to the condition $\alpha_3^* < 0$, where

$$\alpha_3^* = \alpha_3 - \alpha'\alpha_2/\eta_2. \tag{2.5}$$

The instability remains specific to nematics and as a consequence the threshold is again given by a condition of the form (2.2). Now let us summarize the situation in the planar case.

(*a*) When α_3 is negative, the homogeneous instability as well as the roll instability can take place since α_3^* is also negative but the intensity of the mechanisms is different. The type of the instability which occurs can be monitored by a magnetic field applied along the direction of the unperturbed orientation which adds its stabilizing contribution to the elastic restoring torques. In the absence of an external field and under weak fields, the elastic stabilizing contribution is dominant and since the periodic distortion associated with rolls implies a greater elastic expense, the corresponding threshold is higher than for the homogeneous instability. Conversely, under high fields, the magnetic stabilizing contribution is dominant and rolls which correspond to the strongest destabilizing mechanism have the lowest threshold. Experiments have been performed using the well-known nematic compound MBBA (Pieranski & Guyon 1974*a*; Dubois-Violette *et al.* 1977) and the cross-over from the homogeneous instability to the rolls has been found to be in agreement with the sketchy description given above.

(*b*) When α_3 is positive, the homogeneous instability disappears and the rolls can take place as long as α_3 is small enough [see (2.5)]:

$$0 < \alpha_3 < \alpha'\alpha_2/\eta_2.$$

Experimental evidence has been given by Pieranski & Guyon (1976) using CBOOA, a nematic compound for which $\alpha_3 \rightarrow +\infty$ close to a nematic–smectic A phase transition.

In the following we shall examine the particular contribution of rotation to instabilities which are specific to nematics, that is to say instabilities which result from a coupling between orientation fluctuations and occur at a threshold roughly given by

$$s_c \tau_0 \sim 1$$

(for the homogeneous as well as for the roll instability), where τ_0 is the time constant characteristic of the evolution of orientation fluctuations. Before we discuss the effect of rotation, let us specify the geometry of the problem, discuss some approximations and present the linearized hydrodynamic equations.

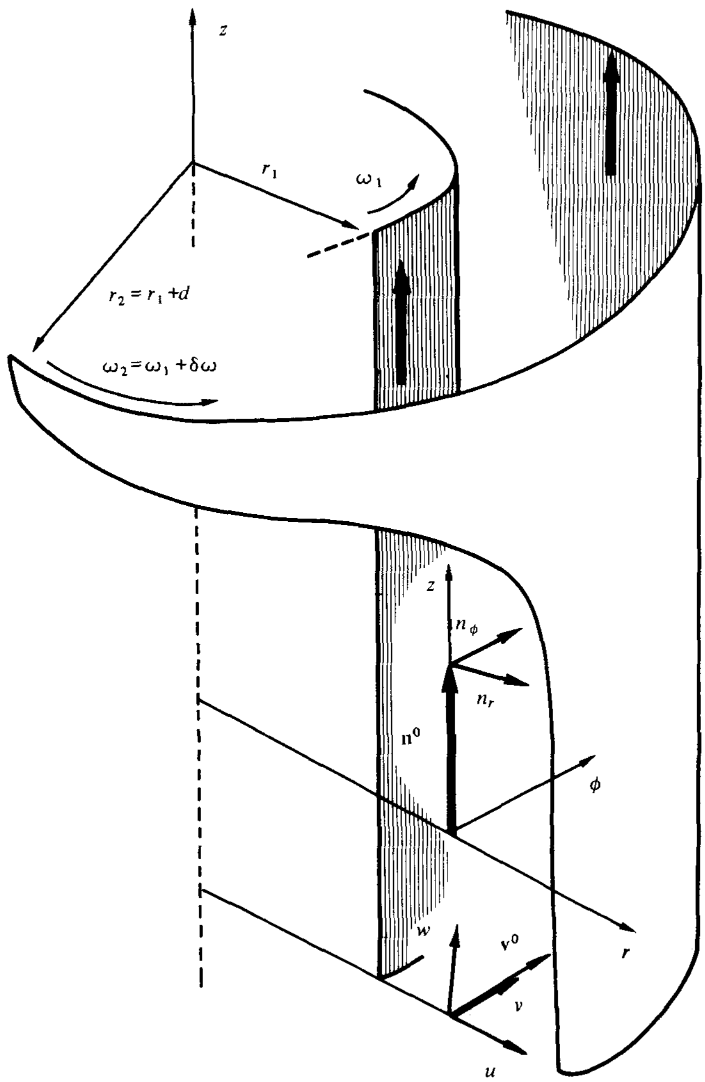

FIGURE 3. Geometry of the cylindrical Couette experiment: ω_1 and ω_2 are the angular velocities of the two cylinders. At the cylinders the director is aligned parallel to the rotation axis. $\mathbf{n}^0$ and $\mathbf{v}^0$ are the unperturbed director and velocity fields; n_r and n_ϕ are the director fluctuations; u, v, w the velocity fluctuations.

3. Linearized hydrodynamic equations

Consider a cylindrical Couette experiment in the geometry of figure 3, where the nematic is enclosed between two long co-axial cylinders, the inner one with radius r_1, rotating at an angular velocity ω_1, and the outer one with radius $r_2 = r_1 + d$, rotating at a velocity $\omega_2 = \omega_1 + \delta\omega$. In addition, we assume that the cylinders have been polished such that the molecular alignment is fixed parallel to the rotation axis Oz (strong anchoring condition); a magnetic field applied parallel to Oz may come and reinforce this boundary effect. Then, in the unperturbed configuration this alignment prevails in the bulk of the nematic and the velocity profile is that obtained for isotropic fluids. In cylindrical co-ordinates (r, ϕ, z) we have

$$n_r^0 = n_\phi^0 = 0, \quad n_z^0 = 1,$$

$$v_r^0 = v_z^0 = 0, \quad v_\phi^0 = Ar + B/r, \tag{3.1}$$

with $$A = (\omega_2 r_2^2 - \omega_1 r_1^2)/(r_2^2 - r_1^2),$$

$$B = (\omega_1 - \omega_2) r_1^2 r_2^2/(r_2^2 - r_1^2).$$

Now let us consider an orientation fluctuation

$$\delta\mathbf{n} = (n_r, n_\phi, 0),$$

a velocity fluctuation $$\delta\mathbf{v} = (u, v, w),$$

and a pressure fluctuation δp.

The complete set of linearized equations governing this infinitesimal perturbation is derived in appendix A within the framework of the Ericksen–Leslie–Parodi hydrodynamic description of nematics.

In order to simplify the mathematical analysis, we shall assume that, in all the following, the gap d is much smaller than the mean radius $r_m = \frac{1}{2}(r_1 + r_2)$. (Since optical observations require thin samples ($d \lesssim 500\,\mu$m) this 'narrow gap approximation' is by no means a restriction unless one uses cylinders as small as those of Cladis & Torza 1975.)

The angular velocity is $$\omega_0(r) = \frac{v_\phi^0}{r} = A + B/r^2,$$

and the shearing rate is $$s(r) = \frac{dv_\phi^0}{dr} - \frac{v_\phi^0}{r} = -\frac{2B}{r^2}.$$

In the narrow gap limit one gets

$$\omega_0(r) = \omega_m + (r - r_m)\,\delta\omega/d,$$

and $$s = r_m\,\delta\omega/d,$$

where ω_m is the mean angular velocity: $\omega_m = \frac{1}{2}(\omega_1 + \omega_2)$. Moreover we have

$$2A = 2\omega_m + s.$$

We shall restrict our attention to the case of axisymmetric fluctuations ($\partial/\partial\phi \equiv 0$) since, at the limit of the planar Couette flow, the distortion which takes place is invariant through a translation in the direction of the flow (Pieranski & Guyon 1974*b*), i.e. the azimuthal direction in the present problem. Then the linearized equations read

(*a*) Torque equations:

$$\Gamma_r = 0 = -\left(K_2 \frac{\partial^2}{\partial r^2} + K_3 \frac{\partial^2}{\partial z^2}\right) n_\phi + \gamma_1 \frac{\partial n_\phi}{\partial t} + \alpha_2 \frac{\partial v}{\partial z} + \alpha_2 s n_r, \tag{3.2}$$

$$\Gamma_\phi = 0 = \left(K_1 \frac{\partial^2}{\partial r^2} + K_3 \frac{\partial^2}{\partial z^2}\right) n_r - \gamma_1 \frac{\partial n_r}{\partial t} - \left(\alpha_2 \frac{\partial u}{\partial z} + \alpha_3 \frac{\partial w}{\partial r}\right) - \alpha_3 s n_\phi. \tag{3.3}$$

These torque equations express the dynamic equilibrium of the director. The Frank elastic contribution to the total torque is easily recognized. The viscous part is the sum of three terms: the orientational viscous damping (viscosity $\gamma_1 = \alpha_3 - \alpha_2$), the contribution of the induced secondary flows (u, v, w) and the direct contribution of the orientation fluctuations (n_r and n_ϕ).

(*b*) Force equations:

$$\rho\left(\frac{\partial u}{\partial t} - 2\omega_0 v\right) = -\frac{\partial(\delta p)}{\partial r} + \left(\eta_2 \frac{\partial^2}{\partial z^2} + \eta'' \frac{\partial^2}{\partial r^2}\right) u + \alpha' s \frac{\partial n_\phi}{\partial z} + \alpha_2 \frac{\partial}{\partial t}\left(\frac{\partial n_r}{\partial z}\right), \tag{3.4}$$

$$\rho\left(\frac{\partial v}{\partial t} + 2Au\right) = \left(\eta_2 \frac{\partial^2}{\partial z^2} + \eta_3 \frac{\partial^2}{\partial r^2}\right) v + (\eta_2 - \eta_3)\, s \frac{\partial n_r}{\partial z} + \alpha_2 \frac{\partial}{\partial t}\left(\frac{\partial n_\phi}{\partial z}\right), \tag{3.5}$$

$$\rho \frac{\partial w}{\partial t} = -\frac{\partial(\delta p)}{\partial z} + \left(\eta' \frac{\partial^2}{\partial z^2} + \eta_1 \frac{\partial^2}{\partial r^2}\right) w + (\eta_1 - \eta_3)\, s \frac{\partial n_\phi}{\partial r} + \alpha_3 \frac{\partial}{\partial t}\left(\frac{\partial n_r}{\partial r}\right). \tag{3.6}$$

These equations express the conservation of linear momentum. On the right-hand side, the viscous forces clearly contain two parts: (i) the anisotropic extension of the usual $\eta\Delta\mathbf{v}$ term present for isotropic liquids and (ii) the viscous contribution linked to a non-uniform orientation.

(*c*) Finally, one must add the usual continuity equation for an incompressible fluid:

$$0 = \frac{\partial u}{\partial r} + \frac{\partial w}{\partial z}. \tag{3.7}$$

At this point, one should develop a conventional 'normal mode analysis' (Chandrasekhar 1961). Instability thresholds and critical wavelengths would be obtained by a calculation parallel to the one Manneville & Dubois-Violette (1976) have performed in the planar shear case. Another point of view has been taken by V. A. Nye (private communication) at Strathclyde University (Glasgow) who follows an approximate method due to Jeffreys (1928). Here we shall rather prefer a semi-quantitative approach which rests on a description of the instability mechanisms in terms of effective torques exerted on molecules. Before we extend the approximate model – already successfully developed for the planar shear case (Manneville & Dubois-Violette 1976; Dubois-Violette *et al.* 1977) – let us briefly review the case of isotropic liquids, which will make the comparison more transparent.

4. Inertial coupling of velocity fluctuations

The simplest description which contains the essentials of the Taylor instability is obtained within the framework of a one-dimensional model which neglects the radial dependence of the fluctuations (a thorough account may be found in Chandrasekhar 1961). The hydrodynamic equations then reduce to

$$\rho\left(\frac{\partial u}{\partial t} - 2\omega_0 v\right) = \eta \frac{\partial^2 u}{\partial z^2}, \tag{4.1}$$

$$\rho\left(\frac{\partial v}{\partial t} + 2Au\right) = \eta \frac{\partial^2 v}{\partial z^2}. \tag{4.2}$$

Let us assume a tangential velocity fluctuation v. It induces a radial Coriolis force

$$F_r^c = 2\rho\omega_0 v.$$

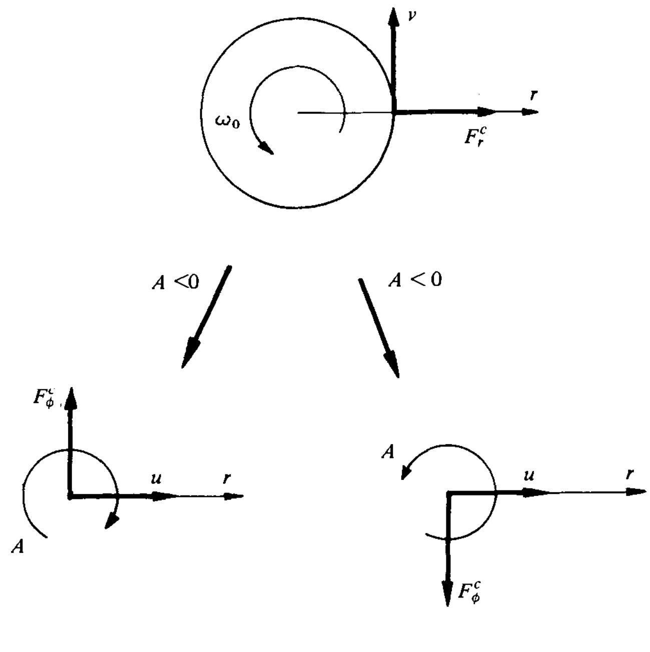

FIGURE 4. Taylor instability mechanism: the Coriclis coupling. Owing to rotation, a radial velocity fluctuation u induces a tangential Coriolis force, the sign of which depends on that of the local vorticity $2A$. Finally, the coupling is stabilizing for $A > 0$. Indeed, a fluctuation $v > 0$ induces a force $F_r^c > 0$ which in turn induces a velocity fluctuation $u > 0$ (*b*); then the tangential force F_ϕ^c tends to damp the initial fluctuation v. Conversely, if $A < 0$, this is destabilizing.

This force tends to create a velocity component u given by

$$\rho \frac{\partial u}{\partial t} = 2\rho\omega_0 v.$$

Now, with u is associated a tangential Coriolis force (figure 4*a*, *b*)

$$F^c = -2\rho A u,$$

which tends to modify the fluctuation v assumed at the beginning. As explained in the caption of figure 4, the mechanism sketched above is locally stabilizing as long as

$$A\omega_0 > 0. \tag{4.3}$$

When $A\omega_0 \leqslant 0$ the mechanism is potentially destabilizing but the instability can take place only when it is strong enough to overcome the stabilizing effect of the diffusion of velocity fluctuations (viscous damping). As to the inviscid fluid, one can see that the fluid is stable when the two cylinders rotate in the same direction (so that ω_0 does not change its sign) – say the positive one – and when

$$A > 0 \quad \text{or} \quad \omega_2 r_2^2 > \omega_1 r_1^2,$$

which corresponds to the Rayleigh criterion applied to the unperturbed velocity profile (3.1). In a real fluid the viscous damping works such that the Taylor instability takes place only when

$$A \leqslant A_c < 0.$$

The order of magnitude of the threshold value A_c may be evaluated assuming $\omega_0 \simeq \omega_m$ (this is consistent with the assumption that the two cylinders rotate in the same direction) and fluctuations of the form

$$(u, v) = (U, V)\sin(q_z z)\exp(\sigma t),$$

with $q_z \sim \pi/d$ corresponding to roughly circular rolls.

The compatibility condition of (4.1) and (4.2) reads

$$(\rho\sigma + \eta q_z^2)^2 + 4\rho^2 A\omega_m = 0,$$

or

$$\sigma_{\pm} = -\frac{\eta}{\rho} q_z^2 \pm (-4A\omega_m)^{\frac{1}{2}}.$$

At the threshold one has $\sigma_+ = 0$ (stationary instability) and

$$4A_c\omega_m = -\left(\frac{\eta}{\rho} q_z^2\right)^2, \tag{4.4}$$

or, using the characteristic time τ_v defined by (2.3),

$$4A_c\omega_m\tau_v^2 = -1. \tag{4.4'}$$

In terms of the Taylor number (see Chandrasekhar 1961)

$$T = -4\rho^2 A\omega_1 d^4/\eta^2,$$

one gets

$$T_c = 2\pi^4/(1+\mu),$$

with $\mu = \omega_2/\omega_1$, in qualitative agreement with the exact result $T_c = 3416/(1+\mu)$ at the limit $\mu \to 1$.

Two parameters ω_1 and ω_2 are at our disposal. Assuming that one of these parameters is kept fixed, the threshold condition (4.4′) gives the critical value of the other one. For example, assuming that the outer cylinder is at rest $\omega_2 = 0$, in the 'narrow gap limit' one gets $2\omega_m = \omega_1$ and $2A = 2\omega_m + s = \omega_1(1 - r_m/d) \simeq -\omega_1 r_m/d$ so that, in terms of ω_1, the critical value is

$$\omega_{1c} = \tau_v^{-1}(d/r_m)^{\frac{1}{2}}.$$

When the two cylinders are rotating, instead of ω_1 and ω_2 one may prefer the mean velocity ω_m and the velocity difference $\delta\omega$ or, even better, ω_m and the shearing rate s in view of the comparison with the nematic case where the instability mechanisms depend on s. Then the threshold condition (4.4′) may be read as an equation giving the critical shearing rate s_c as a function of ω_m:

$$2\omega_m(2\omega_m + s_c)\tau_v^2 = -1,$$

and one can distinguish a 'slow rotation' regime with $2\omega_m\tau_v \ll 1$, where

$$2\omega_m s_c \sim -1/\tau_v^2,$$

from a rapid rotation regime with $2\omega_m\tau_v \gg 1$ and

$$s_c \sim -2\omega_m.$$

This distinction will remain for the nematic case but the behaviour of the threshold s_c will turn out to be drastically different.

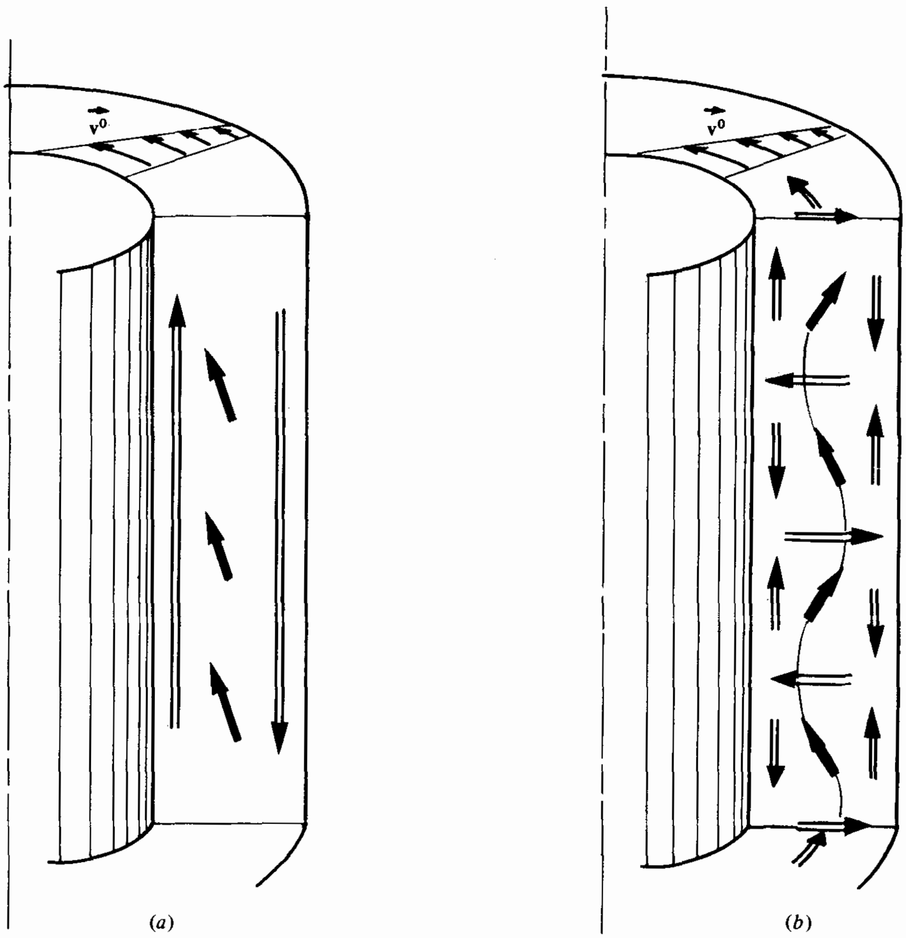

FIGURE 5. Nematic case. (*a*) Homogeneous instability: the secondary flow (⇒) is unaffected by the Coriolis coupling. (*b*) Roll instability: the non-uniform orientation distortion induces secondary flows in the plane perpendicular to the rotation axis to which correspond Coriolis effects as discussed in figure 4.

5. The nematic case in the slow regime

5.1. *Instability mechanism*

Let us now consider the case of nematics and examine first the homogeneous instability which can take place when α_3 is negative. In this instability mode, as explained by Leslie (1976), the only velocity fluctuation which couples to the distortions is the one parallel to the initial director orientation. In the present case this corresponds to a velocity parallel to the rotation axis (figure 5(*a*)), this fluctuation does not involve Coriolis forces and thus we do not expect any rotation effect on the threshold instability. Indeed this may be shown directly from (3.7) and (3.5). Assuming $\partial/\partial z \equiv 0$ and taking into account boundary conditions on u and v, from (3.7), $\partial u/\partial r = 0$, one gets $u = cst = 0$. Then (3.5) simply reads:

$$\rho \frac{\partial v}{\partial t} = \eta_3 \frac{\partial^2 v}{\partial r^2},$$

i.e. v is not coupled with the other variables and damps out; then, since $v = 0$, the only remaining term involving rotation, $2\omega_0 v$, disappears and we are left with the same problem as in planar geometry.†

The roll instability (figure 5 (b)) involves radial and tangential secondary flow. Since these flows are coupled by Coriolis forces as in the case of isotropic fluids, we expect the rotation to affect the threshold. In the limit of the planar shear, the roll instability is stationary – that is to say that the distortion grows without time oscillation according to an exponential law of the form $\exp(\sigma t)$ with σ real. The threshold corresponds to $\sigma = 0$, so we shall assume that the principle of exchange of stabilities also holds in the present problem. This greatly simplifies the equations since one may then forget all terms containing $\partial/\partial t$. Moreover, in this approach we shall again neglect the radial dependence of fluctuations (one-dimensional model) and also, as in the isotropic case, assume that $\omega_0(r) \simeq \omega_m$. Then (3.4) and (3.5) simply read

$$-2\rho\omega_m v = \eta_2 \frac{\partial^2 u}{\partial z^2} + \alpha' s \frac{\partial n_\phi}{\partial z}, \tag{5.1}$$

$$2\rho A u = \eta_2 \frac{\partial^2 v}{\partial z^2} + (\eta_2 - \eta_3) s \frac{\partial n_r}{\partial z}. \tag{5.2}$$

Recalling that orientation fluctuations play the dominant role, we assume

$$(n_r, n_\phi) = (N_r, N_\phi) \cos(q_z z). \tag{5.3}$$

This obviously leads to velocity fluctuations of the form

$$(u, v) = (U, V) \sin(q_z z), \tag{5.4}$$

where

$$U = -\frac{sq_z}{D}(\eta_2 \alpha' q_z^2 N_\phi + 2\rho\omega_m(\eta_2 - \eta_3) N_r), \tag{5.5}$$

$$V = -\frac{sq_z}{D}(\eta_2(\eta_2 - \eta_3) q_z^2 N_r - 2\rho A \alpha' N_\phi), \tag{5.6}$$

$$D = (\eta_2 q_z^2)^2 + 4\rho^2 A \omega_m. \tag{5.7}$$

As explained in § 2, the stability analysis reduces to the study of effective torques exerted on the director. Neglecting the radial dependence of fluctuations the torque equations (3.2) and (3.3) reduce to

$$\Gamma_r = 0 = -K_3 \frac{\partial^2 n_\phi}{\partial z^2} + \alpha_2 \frac{\partial v}{\partial z} + \alpha_2 s n_r, \tag{5.8}$$

$$\Gamma_\phi = 0 = K_3 \frac{\partial^2 n_r}{\partial z^2} - \alpha_2 \frac{\partial u}{\partial z} - \alpha_3 s n_\phi. \tag{5.9}$$

One obtains the effective torque equations by replacing the velocity components u and v of (5.4) in (5.8) and (5.9) using (5.5) and (5.6). They read

$$\left(K_3 q_z^2 + \frac{\alpha_2 \alpha' q_z^2 \, 2\rho A s}{D}\right) N_\phi + \alpha_2 \left(1 - \frac{\eta_2(\eta_2 - \eta_3) q_z^4}{D}\right) s N_r = 0, \tag{5.10}$$

$$-\left(K_3 q_z^2 - \frac{\alpha_2(\eta_2 - \eta_3) q_z^2 \, 2\rho\omega_m s}{D}\right) N_r - \alpha_3 \left(1 - \frac{\eta_2 \alpha_2 \alpha' q_z^4}{\alpha_3 D}\right) s N_\phi = 0. \tag{5.11}$$

† This result does not depend on the narrow gap approximation as can be seen from (A 4) and (A 7) given in appendix A.

One recognizes in the first terms of (5.10) and (5.11) the elastic contributions and Coriolis corrections. The second terms look like those in the planar case: they correspond to a renormalization of the viscosities α_2 and α_3. The threshold expression is obtained by writing the compatibility condition of system (5.10) and (5.11). In the planar case this gives the condition

$$s\tau_0 \sim 1. \tag{5.12}$$

In the present case one will obtain a relation involving both s and ω_m. As s is the pertinent parameter in nematics, one had better consider the threshold as a function $s(\omega_m)$. If one looks at (5.10) and (5.11) one can see that the parameter involving the rotation is $\omega_m \tau_v$, where $\tau_v = \rho/(n_2 q_z^2)$ and that

$$2A\tau_v = (2\omega_m + s)\tau_v = 2\omega_m \tau_v + s\tau_0 \left(\frac{\tau_v}{\tau_0}\right). \tag{5.13}$$

As in nematics $\tau_v/\tau_0 \ll 1$, for shear rates corresponding to the instability threshold $s\tau_0 \sim 1$ one gets

$$2A\tau_v = 2\omega_m \tau_v. \tag{5.14}$$

Expression (5.7) now reads

$$D = (\eta_2 q_z^2)^2 (1 + (2\omega_m \tau_v)^2). \tag{5.15}$$

Clearly two different regimes appear: a slow rotation regime corresponding to $2\omega_m \tau_v \ll 1$ and a fast one for $2\omega_m \tau_v \gg 1$. In this section we shall consider the slow regime ($\omega_m \tau_v \ll 1$) for which (5.14) is valid. (This excludes a very slow regime characterized by $2\omega_m \tau_0 \ll 1$ for which the complete expression (5.13) should be used.) In this limit D reduces to $(\eta_2 q_z^2)^2$. Returning to (5.10) and (5.11) one sees that the second terms are unchanged relative to the planar shear case. The rotation effect only appears in the first terms, which we shall now examine.

Let us first consider the case of a purely radial orientation fluctuation ($N_r \neq 0$, $N_\phi = 0$). Then (5.1) and (5.2) read

$$-2\rho\omega_m V = -\eta_2 q_z^2 U, \tag{5.16}$$

$$2\rho\omega_m U = -\eta_2 q_z^2 V - (\eta_2 - \eta_3) q_z s N_r. \tag{5.17}$$

The limit $\omega_m \tau_v \ll 1$ implies that the left-hand side of (5.17) can be neglected. Then one obtains

$$V = -\frac{1}{q_z}\frac{(\eta_2 - \eta_3)}{\eta_2} s N_r, \tag{5.18}$$

as in the planar case. Now, owing to the Coriolis force, this fluctuation creates a radial velocity fluctuation (absent in the planar case) given by (5.16):

$$U = -\frac{1}{q_z}\frac{(\eta_2 - \eta_3)}{\eta_2} (2\omega_m \tau_v) s N_r. \tag{5.19}$$

This radial flow is not uniform in space and exerts a tangential viscous torque component on the molecules:

$$\Gamma_\phi^c = \alpha_2 \frac{(\eta_2 - \eta_3)}{\eta_2} (2\omega_m \tau_v) s N_r. \tag{5.20}$$

This contribution is destabilizing for usual nematics with $\alpha_2 < 0$ and $\eta_2 > \eta_3$ when $\omega_m s < 0$. For $\omega_m > 0$ this corresponds to the inner cylinder rotating faster than the outer one ($\delta\omega < 0 \Rightarrow \omega_1 > \omega_2$, figure 6).

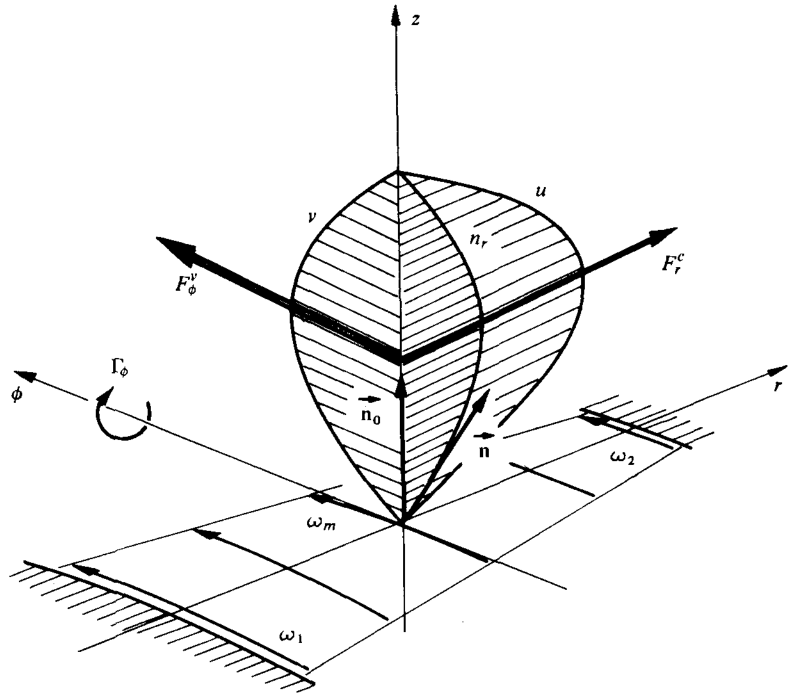

FIGURE 6. Coriolis contribution to the instability mechanism in a nematic: an inhomogeneous radial fluctuation n_r induces a viscous force F^v which creates a velocity fluctuation v. When $\omega_1 > \omega_2$, the Coriolis coupling induces a radial velocity u. The corresponding shear rate $\partial u/\partial z$ tends to increase the initial orientation fluctuation.

A parallel analysis could be performed in the case of a purely tangential orientation fluctuation ($N_\phi \neq 0$, $N_r = 0$) and would lead to the same conclusion. Indeed one would get

$$\Gamma_r^c = \alpha_2 \frac{\partial v}{\partial z} = \frac{\alpha_2 \alpha'}{\eta_2} (2\omega_m \tau_v) sN_\phi. \tag{5.21}$$

This also corresponds to destabilizing torque when $\omega_m s < 0$, since $\alpha' = \frac{1}{2}(\alpha_5 + \alpha_2) < 0$ in general. In order to get an estimate of the threshold one has now to consider the coupling between N_r and N_ϕ as described by the effective torque equations.

5.2. *Threshold*

Let us write the effective torque equations (5.10) and (5.11) in terms of dimensionless quantities:

$$X = s\tau_0 = s|\alpha_2\alpha_3|^{\frac{1}{2}}/K_3 q_z^2, \tag{5.22}$$

$$Y = 2\omega_m \tau_v \ll 1 \tag{5.23}$$

and

$$\lambda = |\alpha_3/\alpha_2|^{\frac{1}{2}}, \quad S = \text{sgn}\{\alpha_3/\alpha_2\},$$

$$\mu = \frac{\alpha_2\alpha'}{|\alpha_3|\eta_2}, \quad \nu = \frac{\eta_2 - \eta_3}{\eta_2},$$

and

$$2l = \lambda\mu + \nu/\lambda = \alpha_2^2/\eta_2|\alpha_2\alpha_3|^{\frac{1}{2}}.$$

α_2 is negative but α_3 may be positive ($S = -1$) as well as negative ($S = +1$); in addition we have

$$\mu > 0 \quad \text{and} \quad 0 < \nu < 1.$$

Using these notations, the effective torque equations read

$$(1+\lambda\mu XY)N_\phi - \frac{1}{\lambda}(1-\nu)XN_r = 0, \tag{5.24}$$

$$\left(1+\frac{\nu XY}{\lambda}\right)N_r - \lambda(S+\mu)XN_\phi = 0, \tag{5.25}$$

and the compatibility condition is

$$(1+\lambda\mu XY)\left(1+\frac{\nu XY}{\lambda}\right) = X^2(1-\nu)(S+\mu),$$

or, after reduction, $$X^2(1-\nu)(S+\mu) - 2lXY - 1 = 0. \tag{5.26}$$

Roots of (5.26) give the critical shearing rate X_c as a function of the overall rotation measured by Y. This equation has two real roots as long as

$$\Delta' = Y^2l^2 + (1-\nu)(S+\mu) > 0.$$

l, ν, μ are of order 1 and since $Y \ll 1$ this implies $S+\mu > 0$. This is always the case when $S = +1$ (i.e. $\alpha_3 < 0$). When α_3 is positive ($S = -1$) this condition implies $\mu > 1$ or

$$\alpha_3 < \alpha'\alpha_2/\eta_2,$$

which is precisely the existence condition for a roll instability in the planar case (§ 2). When this condition is fulfilled, roots of (5.26) are easily found:

$$X_{c\pm}(Y) = \pm\frac{1}{(1-\nu)^{\frac{1}{2}}(S+\mu)^{\frac{1}{2}}} + \frac{lY}{(1-\nu)(S+\mu)}. \tag{5.27}$$

The threshold variation is proportional to $Y = 2\omega_m \tau_v$ and the proportionality coefficient $l/(1-\nu)(S+\mu)$ is of the order of 1. In agreement with the analysis of mechanisms, it may be checked that, when the outer cylinder rotates faster than the inner one, one has $s > 0$ (with $\omega_m > 0$) or $X > 0$ (with $Y > 0$) and the rotation has a stabilizing effect:

$$0 < X_{c+}(0) < X_{c+}(Y);$$

when the inner cylinder rotates faster one has $X < 0$ and the destabilizing contribution of rotation corresponds to a lowering of the threshold:

$$X_{c-}(0) = -X_+(0) < X_{c-}(Y) < 0,$$

that is to say $$|X_{c-}(Y)| < |X_{c-}(0)|.$$

Remark 1: *Ultra-slow regime.* In contrast to the case just examined, in the ultra-slow regime one cannot neglect s compared with ω_m. This leads to modifications as is illustrated by the case where the inner cylinder is at rest. Then $s = 2\omega_m r_m/d$,

$$2A = s \sim \tau_0^{-1} \quad \text{(for an instability specific to nematics)},$$

$$D \sim (\eta_2 q_z^2)\left(1+\frac{d}{r_m}\left(\frac{\tau_v}{\tau_0}\right)^2\right) \simeq (\eta_2 q_z)^2.$$

The terms involving the rotation disappear completely as one can see in the effective torque equations (5.10) and (5.11). For example, let us just explicate the form of the first term of (5.10):

$$K_3 q_z^2 + \frac{\alpha_2 \alpha' q_z^2\, 2\rho A s}{D} \simeq K_3 q_z^2(1+s^2\tau_v\tau_0) \simeq K_3 q_z^2.$$

Interpretation of the experiments by Dubois-Violette *et al.* (1977) is then completely justified despite the use of a cylindrical geometry (however, notice that, if corrections due to rotation were perfectly negligible, corrections linked to curvature should have been taken into account since the aspect ratio $d/r_m \simeq 0{\cdot}16$ was not small enough to justify a narrow gap approximation).

In this ultra-slow regime, $\omega_0(r)$ cannot be considered as a constant since $\delta\omega$ is of order ω_m. But the effects of rotation are negligible. In all other cases ω_0 can be taken as a constant since now

$$\delta\omega = \frac{d}{r_m}s \sim \frac{d}{r_m}\tau_0^{-1} \ll \tau_0^{-1} < 2\omega_m.$$

Remark 2: *Limit of the one-dimensional model.* From a dimensional point of view the wave vector q_z of the unstable mode must be related to the gap d through the boundary conditions imposed at the cylinders. The model developed so far implicitly assumes $q_z \sim \pi/d$ but roughly constant.

In order to get the true spatial dependence of the instability mode, one needs a bi-dimensional analysis. In particular, in the case of the fast rotation regime this one-dimensional analysis would be misleading since the wavelength dependence will appear to be crucial.

6. Bi-dimensional analysis

The simplified approach developed above must now be completed to take into account the radial dependence of the fluctuations and the associated boundary conditions. Such an analysis has already been performed in the case of the planar Couette flow. With regard to the homogeneous instability which can take place when α_3 is negative and which is not modified by rotation effects, in the narrow gap limit, the threshold will have the same value as in the planar case (Manneville & Dubois-Violette 1976; Leslie 1976). Using the definition

$$X = s\tau_0 = s\left(\frac{d}{\pi}\right)^2\left(\frac{|\alpha_2\alpha_3|}{K_1K_2}\right)^{\frac{1}{2}}, \tag{6.1}$$

which makes an explicit reference to the gap d, we have

$$X_c = \pm\, 0{\cdot}936,$$

where the $\pm$ sign depends on which cylinder rotates faster. In fact, the threshold weakly depends on the ratio η_1/η_3. The value reported above corresponds to MBBA at 25 °C.

For the roll instability the exact 'normal mode analysis' is far more tedious. Happily an 'approximate' normal mode analysis has been shown to lead to excellent quantitative agreement for the plane Couette flow (Dubois-Violette *et al.* 1977). This approximate analysis essentially assumes a simplified form for the unstable normal mode and leads to a discussion in terms of effective torques. For the present problem, the simplified normal mode reads

$$(n_r, n_\phi) = (N_r, N_\phi)\cos(q_r(r-r_m))\cos(q_z z)\exp(\sigma t), \tag{6.2}$$

where $q_r = \pi/d$ in order to fulfil the boundary conditions $n_r = n_\phi = 0$ imposed at the cylinders $(r = r_m \pm \frac{1}{2}d)$. The stationary instability corresponds to $\mathrm{Re}\,\sigma = 0$. The wave vector q_z is left free to adjust to the optimum value resulting from the competition between the destabilizing viscous torques and the stabilizing elastic ones.

It can be checked that (6.2) leads to a radial velocity profile

$$u = U \sin(q_r(r-r_m))\cos(q_z z),$$

which does not fulfil the boundary condition $u = 0$ at the cylinders. In fact, the exact distortion profile is a superposition of five modes q_r^j such that

$$n_\phi = \sum_{j=1} N_\phi^j \cos(q_r^j(r-r_m))\cos(q_z z).$$

Among these modes, one (say q_r^1) is very close to π/d and roughly gives the aspect of the distortion profile ($j \neq 1 \Rightarrow N_\phi^j \ll N_\phi^1$) apart from a narrow layer close to the walls where the role of the other modes becomes important in order to fulfil all boundary conditions and, more especially, the one relative to the radial velocity u (see figure 11 in Manneville & Dubois-Violette 1976). The existence of this dominant wave vector $q_r^1 \sim \pi/d$ is another facet of the dominant role of the orientation fluctuations n_r and n_ϕ which have a prior right for the fulfilment of boundary conditions, so that assumption (6.2) turns out to be quantitatively well justified.

As we have seen above, except in an ultra-slow rotation regime (to which belongs the case with one cylinder at rest) where rotation effects are completely negligible, we can neglect the radial dependence of $\omega_0(r)$ and merely assume $\omega_0(r) \simeq \omega_m$. Then all the coefficients of the differential system are constant. The extension of the approximate model to the cylindrical case is then straightforward and should lead to quantitative results for the roll instability.

The sequel of the discussion consists of (*a*) a derivation of effective torque equations from (3.2) and (3.3) through the elimination of all variables except n_r and n_ϕ as given by (6.2) and (*b*) the detailed examination of the corresponding compatibility condition which will give the critical wave vector q_z and the threshold s_c as functions of ω_m.

Effective torque equations are given in appendix B, and for the stationary instability ($\sigma = 0$) they read

$$(f_\phi - 2As\epsilon)N_\phi + (\alpha_2^* s)N_r = 0,$$

$$(f_r - 2\omega_m s\epsilon'')N_r + (\alpha_3^* s)N_\phi = 0.$$

The compatibility condition is

$$\alpha_2^* \alpha_3^* s^2 = (f_\phi - 2\omega_m s\epsilon)(f_r - 2\omega_m s\epsilon''). \tag{6.3}$$

In addition to X given by (6.1) we shall define

$$Y = 2\omega_m \frac{\rho}{\eta_2}\left(\frac{d}{\pi}\right)^2,$$

and we shall measure wave vectors in units of $q_r = \pi/d$:

$$Q = q_z/q_r = dq_z/\pi.$$

Equation (6.3) may be written as a relation between X, Y and Q of the form

$$A(Y,Q)X^2 + B(Y,Q)X + C(Y,Q) = 0,$$

which generalizes (5.26). Y being kept fixed, the threshold corresponds to the minimum of X as a function of Q. In general this equation will have two real roots $X_\pm$ and we

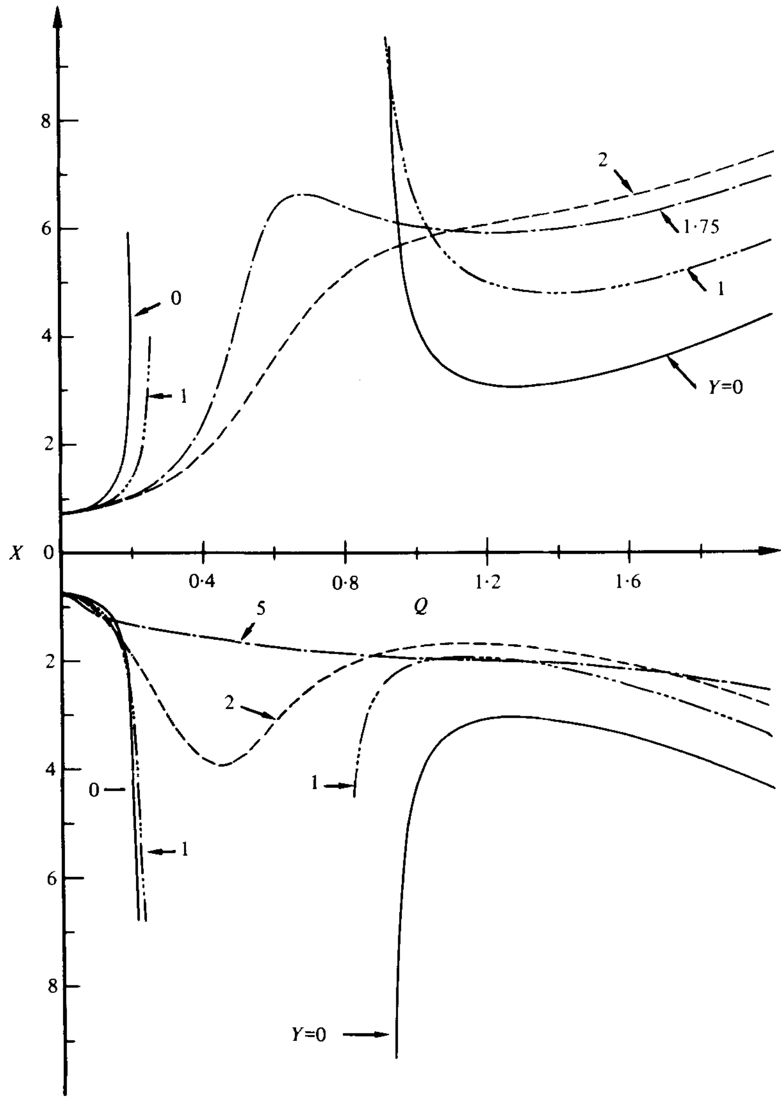

FIGURE 7. Critical curves for different (reduced) rotation rates Y when $\alpha_3 < 0$. X is the reduced shearing rate and Q the reduced wave vector, the minimum at $Q = 0$ corresponds to the homogeneous instability. The threshold of the roll instability corresponds to the minimum of X as a function of $Q \neq 0$.

shall get two thresholds, one for $X_+ > 0$ the other for $X_- < 0$ corresponding to the minimum of $|X_\pm|$. The qualitative analysis suggests separating the cases with α_3 positive or negative. So we shall study first the case $\alpha_3 < 0$ and more particularly the application to MBBA.† Afterwards, we shall turn to the case $\alpha_3 > 0$. Since it is rather difficult to get the complete set of viscoelastic constants for a nematic of this kind, we prefer to limit ourselves to the case α_3 small and perform the calculation for a fictitious nematic having the same viscoelastic constants as MBBA except α_3, which changes

† We shall use viscosities given by Gähwiller (1973).

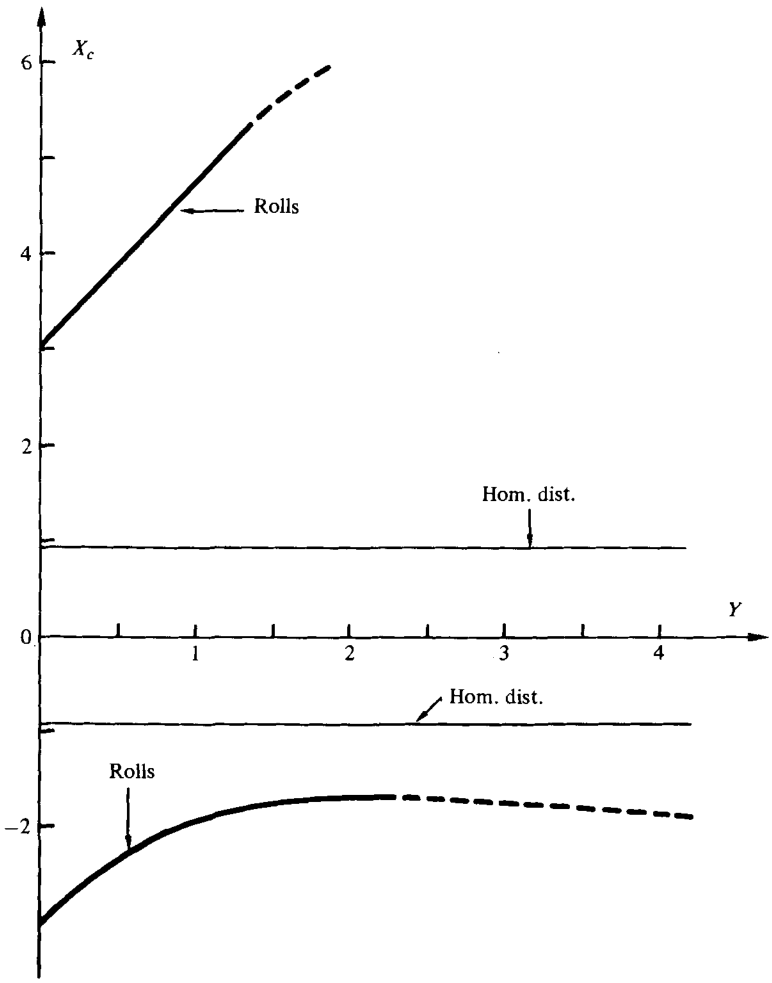

FIGURE 8. Threshold X_c of the roll instability as a function of the rotation rate Y for $\alpha_3 < 0$.

sign. In fact α_3 is linked to the other viscosity coefficients by an Onsager relation (Parodi 1970) but since α_3 remains very small we shall neglect the effect of the sign change on the other viscosities.

6.1. *Effects of the rotation when α_3 is negative*

When α_3 is negative the homogeneous instability takes place at a critical shearing rate which is not modified by the rotation. In figure 7, where X is plotted versus Q for different Y values, this corresponds to the point at $Q = 0$. However, the threshold is not well evaluated by the approximate model and one should prefer the exact value reported above ($X_{c\pm} = \pm 0{\cdot}936$). Anyway, it corresponds to a branch of curve defined for $Q < 1$. In contrast the roll instability corresponds to the minimum of a second branch defined for $Q > 1$. As long as Y is small enough these two branches are well separated by a region where α_3^* given by (B 9) is positive (i.e. where the fundamental instability condition $\alpha_3^* < 0$ is not fulfilled). When Y increases, the separation region

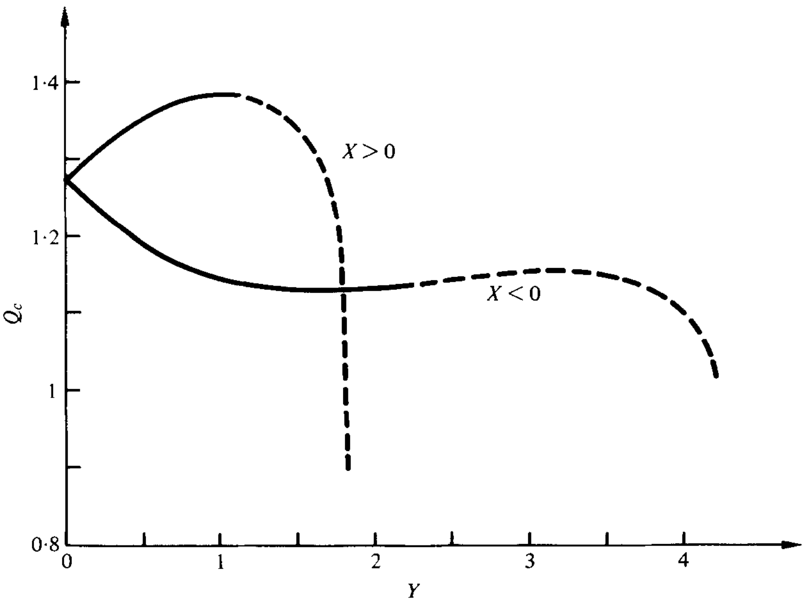

FIGURE 9. Critical wave vector Q_c as a function of the rotation rate Y for $\alpha_3 < 0$.

disappears ($Y > 1{\cdot}7$), the minimum corresponding to rolls becomes less and less pronounced and disappears completely at $Y \sim 1{\cdot}8$ for X_+ and at $Y \sim 4$ for X_-. Considering the result of the exact bi-dimensional analysis obtained in the plane shear case (Manneville & Dubois-Violette 1976) one may question the quantitative character of the approximate normal mode analysis when the minimum at $Q > 1$ is no longer pronounced. The description of the rapid rotation regime is then not very reliable but should be qualitatively correct and a roll instability is not to be expected for $Y \gg 1$. In figures 8 and 9, we have plotted the thresholds $X_{c\pm}$ and the critical wave vectors Q_c of the rolls as functions of the mean rotation measured by Y. The part of the curves drawn with dashes should be confirmed by an exact calculation.

In the slow rotation regime Q_c does not vary much (figure 9), which explains the agreement with the qualitative analysis given in § 5. From figure 8 we deduce that the homogeneous instability represented by the straight horizontal lines $X_c = \pm 0{\cdot}936$ has always the lowest critical value, so that we never expect to observe the roll instability when increasing the overall rotation ω_m. This situation should remain in the presence of a small stabilizing field applied parallel to the rotation axis Oz. However, in the planar shear case a cross-over from the homogeneous instability to the rolls is expected when the field strength increases. In cylindrical geometry the addition of an overall rotation will then shift the value of the cross-over field H_c, decreasing H_c when X is negative (destabilizing effect which favours the rolls), increasing it in the opposite case.

6.2. *Effect of rotation when α_3 is positive and small*

When α_3 is positive the branch defined for $Q \lesssim 1$ disappears and one is left with the branch corresponding to rolls as can be seen on figure 10. It must be noticed that, when

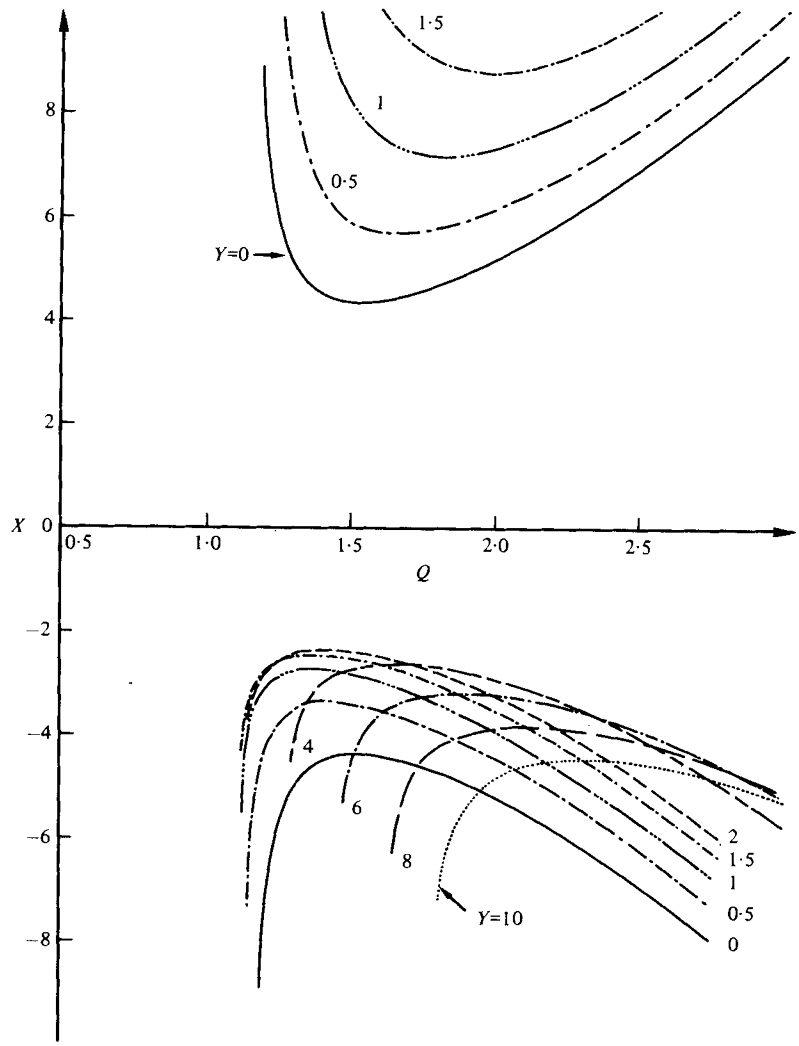

FIGURE 10. Critical curves for different rotation rates Y when $\alpha_3 > 0$. Only rolls can develop.

Y increases, the domain where this branch is defined is pushed towards large values of Q. This may be easily understood from an expansion of α_3^* given by (B 9) retaining only leading terms in Q^2. The condition $\alpha_3^* < 0$ may be written as

$$\mu Q^4/(Q^4 + Y^2) > 1, \tag{6.4}$$

where $$\mu = \alpha_2 \alpha'/\eta_2 \alpha_3,$$

(6.4) also reads $$(\mu - 1) Q^4 > Y^2, \tag{6.5}$$

which implies $\mu > 1$ or $\alpha_3 < \alpha_{3l}$, $\alpha_{3l} = \alpha_2 \alpha'/\eta_2$, i.e. the existence condition for instabilities specific to nematics in the planar case.

Numerical results are given in figures 11 and 12, where one can recognize a slow rotation regime analogous to the one for true MBBA (with $\alpha_3 < 0$). For the rapid

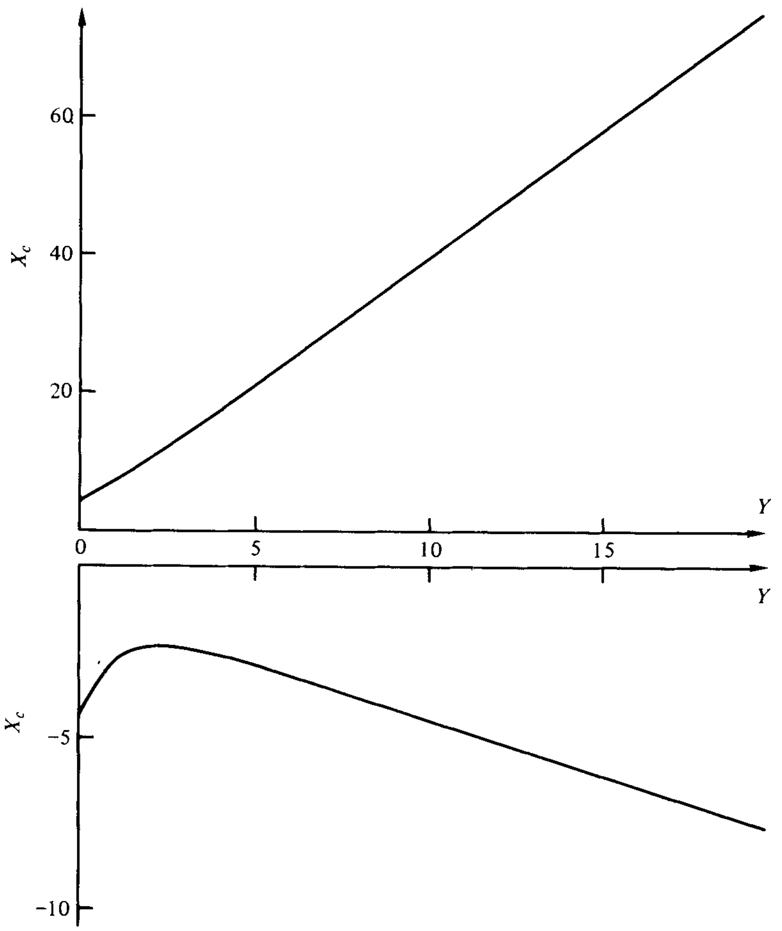

FIGURE 11. Threshold X_c as a function of the rotation rate Y for $\alpha_3 > 0$. Notice that, taking into account the scale change, the slope of the two curves are the same at $Y = 0$.

rotation regime $Y > 1$ the minimum is well pronounced (contrary to the $\alpha_3 < 0$ case) and the approximate analysis is satisfactory. As we could have inferred from (6.5), an expansion of the compatibility condition leads to the asymptotic regime

$$|X_c| \propto Y, \tag{6.6}$$

with

$$Q_c \propto Y^{\frac{1}{2}}, \tag{6.7}$$

where the proportionality constants are of the order of 1 but different for positive and negative solutions. As to this regime, another difference from the case of isotropic liquids may be pointed out. In this latter case, the rapid rotation regime corresponds to $s_c = -2\omega_m$ or with the present notations

$$X_c = s_c \tau_0 = -2\omega_m \tau_0 = -2\omega_m \tau_v \frac{\tau_0}{\tau_v}$$

$$= -\frac{\tau_0}{\tau_v} Y \propto Y,$$

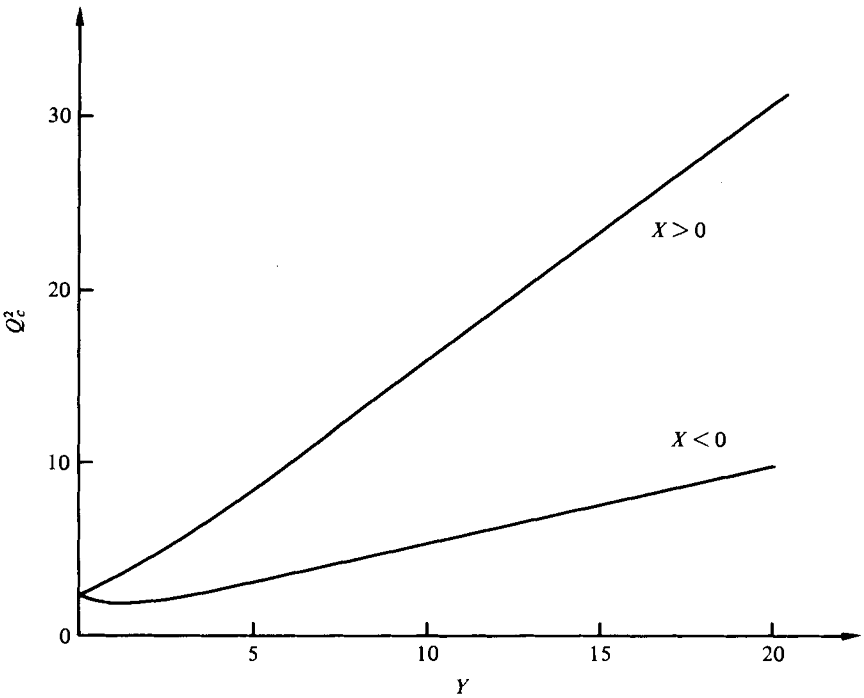

FIGURE 12. Critical wave vector Q_c as a function of the rotation rate Y for $\alpha_3 > 0$.

but this time the proportionality constant is very large, of the order of 10^5. Moreover the Taylor instability takes place only when X is negative (inner cylinder rotating faster) contrary to the present case where the instability specific to nematics takes place even when the outer cylinder rotates faster ($X > 0$). From the point of view of the mechanisms, one would like to have a simple picture of the asymptotic regime (6.6), (6.7) as was the case for the slow rotation regime (§5). However, this seems difficult to obtain since the threshold value derives from a delicate balance between different contributions where the value of the wave vector plays a crucial role.

7. Oscillatory instability

When α_3 is positive it has been shown in §6.2 [equations (6.4)–(6.5)] that the condition for a stationary instability is

$$\alpha_3 < \alpha_{3l}.$$

We now consider the case $\alpha_3 > \alpha_{3l}$ and examine the possibility of an oscillatory instability. Thus we look for fluctuations of the form (6.2) with $\sigma = \sigma' + i\sigma''$ and $\sigma'' \neq 0$, $\sigma' = 0$ corresponding to the threshold. Force and torque equations are given in appendix B.

From a general point of view one can say that when the basic mechanisms are stabilizing for a stationary regime they can nevertheless induce global destabilizing effects if there are adequate phase-shifts between the different contributions. A simple and illustrative example is given by Lekkerkerker (1977) in the case of thermal instabilities in nematics. Here, for example, looking at the radial force equation (B 3) one sees that now the contribution of the two fluctuations n_ϕ and n_r have a phase lag of $\frac{1}{2}\pi$ and that v is in phase with n_ϕ. Returning to the radial torque component (B 1) it appears that an instability can take place if the stabilizing elastic contribution due to n_ϕ, in phase with n_r, is cancelled by the viscous destabilizing one due to the velocity fluctuation v (induced by the Coriolis effect).

We shall not go into details but just give the two effective torque equations obtained after elimination of p, u, v, w between the force and torque equations (B 1)–(B 6). At threshold ($\sigma' = 0$) they read

$$(i\gamma_\phi \sigma'' + f_\phi - 2\omega_m s\epsilon) N_\phi + (\alpha_2^* s + 2i\omega_m \sigma'' \epsilon') N_r = 0, \tag{7.1}$$

$$(i\gamma_r \sigma'' + f_r - 2\omega_m s\epsilon'') N_r + (\alpha_3^* s - 2i\omega_m \sigma'' \epsilon') N_\phi = 0, \tag{7.2}$$

where the different parameters are defined in appendix B. In these expressions we have set $2A = 2\omega_m$; this excludes the case of the ultra-slow regime.

The compatibility condition of (7.1) and (7.2) reads

$$-(\gamma_r \gamma_\phi + 4\omega_m^2 \epsilon'^2) \sigma''^2 + i\sigma''(\gamma_r f_\phi + \gamma_\phi f_r - 2\omega_m s(\epsilon\gamma_r + \epsilon''\gamma_\phi + \epsilon'(\alpha_3^* - \alpha_2^*)) + (f_r - 2\omega_m s\epsilon'')(f_\phi - 2\omega_m s\epsilon) - \alpha_2^* \alpha_3^* s = 0. \tag{7.3}$$

Since σ'' is real this implies that the imaginary terms of (7.3) are zero. This gives the critical condition

$$\gamma_r f_\phi + \gamma_\phi f_r = 2\omega_m s(\epsilon\gamma_r + \epsilon''\gamma_\phi + \epsilon'(\alpha_3^* - \alpha_2^*)).$$

The expression of the right-hand side may be simplified to give

$$2\omega_m s = \frac{D(\gamma_r f_\phi + \gamma_\phi f_r)}{\rho\gamma_1(\eta_2 - \eta_3)(\alpha_{3l}' - \alpha_3) q_z^2 q_r^2}, \tag{7.4}$$

where

$$\alpha_{3l}' = \frac{\alpha_2 \alpha'}{(\eta_2 - \eta_3)} > \alpha_{3l}.$$

Equation (7.4) defines the critical condition for the oscillatory instability in the same way as (6.3) for the stationary case. The numerator of (7.4) is always > 0. In fact it is obvious in equations (B 10), (B 11) and (B 12) that $D, f_\phi, f_r > 0$. As to γ_ϕ one has

$$\gamma_\phi(\omega) > \gamma_\phi(\omega = 0) = \gamma_1 - \frac{\alpha_2^2 q_z^2}{\eta_2 q_z^2 + \eta_3 q_r^2} = \frac{\gamma_1 \eta_3 q_r^2 + (\gamma_1 \eta_2 - \alpha_2^2) q_z^2}{\eta_2 q_z^2 + \eta_3 q_r^2} > 0,$$

since one recognizes the bend viscosity $\eta_B = \gamma_1 - (\alpha_2^2/\eta_2)$, which by virtue of thermodynamic inequalities (Leslie 1968) is always positive. The case of γ_r is similar. Depending on the value of the viscosity coefficient α_3 one obtains an oscillatory instability with $s > 0$ (outer cylinder rotating faster than the inner one) for $\alpha_3 < \alpha_{3l}'$ and $s < 0$ for $\alpha_3 > \alpha_{3l}'$. The threshold value corresponds to the minimum of s as a function of q_z. Using the dimensionless parameters defined in § 6 it is defined by

$$X = \frac{\tilde{\eta}}{(\alpha_{3l}' - \alpha_3)} \frac{P(Q^2, Y^2)}{YQ^2}, \tag{7.5}$$

$$\frac{\partial X}{\partial Q} = 0, \tag{7.6}$$

where $\tilde{\eta}$ is an effective viscosity and P is a polynomial of the fourth degree in Q^2 of the following form:

$$L_0 + (L_1 + M_1 Y^2) Q^2 + (L_2 + M_2 Y^2) Q^4 + L_3 Q^6 + L_4 Q^8,$$

where the different coefficients depend on the value of the viscoelastic constants. To predict a precise value of the threshold one should know the exact experimental values of the viscosity. Nevertheless, one can deduce scaling laws for the threshold. Equation (7.6) defines the critical wave vector Q_c and, in the limit of slow rotation ($Y \ll 1$), Q_c is roughly independent of Y. The threshold is given by

$$X_c = \frac{\tilde{\eta}}{(\alpha'_{3l} - \alpha_3)} \frac{P(Q_c^2, 0)}{Y Q_c^2}.$$

On the other hand, in the limit of high rotation ($Y \gg 1$) one gets

$$Q_c^2 \sim \frac{1}{Y},$$

and

$$X_c \sim \frac{Y}{(\alpha'_{3l} - \alpha_3)}.$$

Once the threshold value is known one can obtain the period of the oscillations from the real part of (7.3) which relates σ'' to ω_m (or Y), s_c (or X_c), and q_{zc} (or Q_c). It turns out that $\sigma'' \sim s_c$ and in both cases $Y \gg$ or $\ll 1$, $\sigma'' \tau_v \ll 1$, which justifies the neglect of the inertial terms $\rho(\partial v_x/\partial t)$ in the force equations.

8. Conclusion

In this paper we have studied the stability of the Couette flow of nematics when the unperturbed orientation is parallel to the rotation axis. This problem is quite specific to nematics and is very far from the classical Taylor problem in isotropic liquids. Indeed the main destabilizing mechanism results from a constructive coupling between fluctuation orientations via the viscous torques. The relevant flow parameter is the shearing rate and the threshold is roughly given by

$$X_c = s_c \tau_0 \sim 1,$$

where τ_0 is the relaxation time for orientation fluctuations over distances of the order of the gap d:

$$\tau_0 = \gamma d^2 / K\pi^2.$$

Optical observations require thin fluid films ($d \lesssim 500\,\mu$m); for a typical nematic with $\gamma \sim 0{\cdot}1$ to 1 P and $K \sim 10^{-6}$ dyne, this leads to quite large τ_0, typically of order 1 to 10 s for our problem, and accordingly to very low thresholds. In the case of isotropic liquids, the destabilizing mechanism involves an inertial coupling between velocity fluctuations via the Coriolis forces. Then the relevant parameter is, rather, a mean rotation rate ω_m and the threshold is very roughly given by

$$Y_c = 2\omega_m^c \tau_v \sim 1,$$

where τ_v is the relaxation time for velocity fluctuations:

$$\tau_v = \rho d^2 / \eta \pi^2.$$

In the same experimental conditions as before, this corresponds to about 10^{-2} s and consequently to high rotation rates $\omega_m \sim 10^2\,s^{-1}$ or 10 cycles/s.

In the present problem the effect of Coriolis forces appears coupled to specific mechanisms of nematics as is summarized in the following sequence:

$$\text{Orientation fl.} \rightarrow \text{viscous force} \rightarrow \text{velocity fl.} \rightarrow \text{Coriolis force} \rightarrow$$
$$\text{velocity fl.} \rightarrow \text{viscous torque} \rightarrow \text{orientation fl.}$$

To evaluate the effect of the rotation the proper parameter remains $Y = 2\omega_m \tau_v$.

This leads us to define a slow rotation regime ($Y < 1$) and a rapid one ($Y > 1$). However, owing to the order of magnitude of τ_v, the rapid rotation regime seems difficult to achieve. So the detailed mechanism has been examined for a slow rotation. It turns out to be destabilizing when the product $2\omega_m s$ is negative. Several different cases may be distinguished according to the sign and the precise value of the viscosity coefficient α_3.

First of all the inertial (Coriolis) coupling of velocity fluctuations only affects the rolls and not the homogeneous instability (which can only take place when α_3 is negative). The corresponding homogeneous instability threshold is then constant. At least, for MBBA and in the absence of external fields, the homogeneous instability is expected at all rotation rates. Under high enough magnetic field one should observe a change of the cross-over field between the homogeneous instability and the rolls. The case with $\alpha_3 > 0$ seems equally promising. The homogeneous instability no longer takes place but the rolls still remain as long as α_3 is small enough and the modification of the threshold by rotation should be observable.

Now one should notice that the variation of the cross-over field H_c when α_3 is negative as well as the variation of the threshold when α_3 is positive is very small unless high enough mean rotation rates are achieved. First, when one of the cylinders is at rest corrections are of the order of $\tau_v/\tau_0 \sim 10^{-5}$ and then completely negligible. Outside this 'ultra-slow rotation regime' corrections remain usually small. Indeed in the 'slow rotation regime' the correction to the threshold Δs due to the rotation is scaled as

$$\frac{\Delta s}{|s_c|} \sim 2\omega_m \tau_v,$$

and $\Delta s/|s_c| \sim 10\,\%$ already requires a high mean rotation (of the order of 1 cycle/s). The 'rapid rotation regime' involves very high rotation rates and so the asymptotic regime when α_3 is positive seems unlikely to be observed.

In isotropic liquids the Coriolis coupling is responsible for the Taylor instability. In nematics we have studied the modification of the threshold of an instability which precedes in the same way as one studies the effect of a rigid-body rotation on the Bénard instability in isotropic liquids (see Chandrasekhar 1961). Indeed everything happens as if one superimposes a large rigid-body rotation at a rate ω_m ($\lesssim \tau_v^{-1}$) on a small differential rotation $\delta\omega = sd/r_m$ ($\sim (d/r_m)\tau_0^{-1}$) responsible for the instability. Of course since the rotation effect is proportional to $2\omega_m s$, the result is sensitive to the direction of the overall rotation. When the external cylinder rotates faster the mechanism is stabilizing and the threshold increases; when the situation is reversed the threshold decreases but the instability exists in the two cases (which is again different from the Taylor case).

When α_3 is positive and large, the Pieranski–Guyon mechanism is stabilizing even for rolls. A first qualitative analysis in terms of effective torques suggests that an oscillatory (roll) instability is possible; the threshold corresponds to a balance between elastic restoring torques and destabilizing Coriolis contributions. The oscillatory character comes from the over-stabilizing effect of the Pieranski–Guyon mechanism.

For the moment, experiments performed in cylindrical geometry concern either the present problem (**n** parallel to the rotation axis) but in a case where rotation effects are completely negligible (Dubois-Violette *et al.* 1977) or another orientation configuration. In particular, Cladis & Torza have examined the case of a nematic with $\alpha_3 > 0$ and homeotropic boundary conditions. Contrary to appearances, our present study may not be completely irrelevant since they have observed several intermediate configurations before a well-organized 'Taylor' roll pattern. As one of these intermediate states, we may expect a situation where molecules are parallel to the rotation axis everywhere except in a narrow layer close to the cylinders. This situation would result from a Pikin instability (Pieranski, Guyon & Pikin 1976) and would be quite compatible with the 'isotropic-like' velocity profile they presented in their 1975 paper. So our results might be extended to this case.

Anyway, our first results lead to predictions which ask for an experimental check for the stationary instability as well as for the oscillatory instability.

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Appendix A

In orthogonal curvi-linear co-ordinates and under co-variant form,† the Ericksen–Leslie–Parodi equations of nemato-dynamics read:

(i) *Acceleration equation*

$$\rho\frac{dv^i}{dt} = -\delta^{ji}p,j + \sigma^{ji},j \quad \text{where} \quad \frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}.\nabla,$$

and with
$$\sigma^{ji} = \sigma^{ji(e)} + \sigma^{ji(v)}.$$

The reversible part $\sigma^{ji(e)}$ is given by

$$\sigma^{ji(e)} = -\left(\frac{\partial F}{\partial n_{k,j}}\right)(n_{k,l}\delta^{li}),$$

where
$$F = \tfrac{1}{2}\{K_1(\operatorname{div}\mathbf{n})^2 + K_2(\mathbf{n}.\operatorname{curl}\mathbf{n})^2 + K_3(\mathbf{n}\times\operatorname{curl}\mathbf{n})^2\}$$

is the Frank elastic energy (Frank 1958). The viscous part $\sigma^{ji(v)}$ is given by the Leslie stress tensor (Leslie 1968)

$$\sigma^{ji(v)} = \alpha_4 A^{ji} + \alpha_1 n^j n^i(n_k A^{kl} n_l) + \alpha_2 n^j N^i + \alpha_3 N^j n^i + \alpha_5 n^j(n_k A^{ki}) + \alpha_6(n_k A^{kj})\, n^i,$$

with
$$A_{ji} = \tfrac{1}{2}(v_{i,j} + v_{j,i}),$$

and
$$\mathbf{N} = \frac{d\mathbf{n}}{dt} - \tfrac{1}{2}(\operatorname{curl}\mathbf{v})\times\mathbf{n}.$$

† V_i and V^i respectively denote the co- and contra-variant components of a vector $V.f, i$ denotes the co-variant derivative relative to the curvi-linear co-ordinate ξ_i.

(ii) *Angular momentum conservation* leads to a balance of torques exerted on the molecules

$$\mathbf{\Gamma}^{(t)} = \mathbf{\Gamma}^{(e)} + \mathbf{\Gamma}^{(v)} = 0.$$

The elastic part $\mathbf{\Gamma}^{(e)}$ is given by $\quad \mathbf{\Gamma}^{(e)} = \mathbf{n} \times \mathbf{h},$

where the 'molecular field' components read

$$h^i = -\left\{\frac{\partial F}{\partial n_i} - \left(\frac{\partial F}{\partial n_{i,j}}\right)_{,j}\right\}.$$

The viscous part $\mathbf{\Gamma}^{(v)}$ is related to the antisymmetric part of the Leslie stress tensor $\bar{\sigma}^{(v)}$:

$$\mathbf{\Gamma}^{(v)} = -\mathbf{n} \times (\gamma_1 \mathbf{N} + \gamma_2 \bar{\bar{A}} \,.\, \mathbf{n})$$

with $\gamma_1 = \alpha_3 - \alpha_2$ and $\gamma_2 = \alpha_6 - \alpha_5 = \alpha_3 + \alpha_2$ (Parodi 1970).

The continuity equation for any incompressible fluid reads

$$\operatorname{div} \mathbf{v} = 0.$$

In the following, we shall write equations governing the physical components of a fluctuation in cylindrical co-ordinates. The velocity fluctuation components will be u, v and w, the pressure δp and the director fluctuation component n_r and n_ϕ (not to be confused with the previous notation for co-variant quantities). Zeroth order gives the unperturbed quantities:

$$0 = \frac{\partial^2 v^0}{\partial r^2} + \frac{1}{r}\frac{\partial v^0_\phi}{\partial r} - \frac{v^0_\phi}{r^2}$$

leads to

$$v^0_\phi = Ar + \frac{B}{r}, \tag{A 1}$$

and

$$\rho \frac{v^0_\phi}{r} = \frac{\partial p^0}{\partial r}$$

to the unperturbed pressure.

Let us denote

$$\mathscr{D} = \frac{\partial}{\partial r}; \quad \mathscr{D}_* = \frac{\partial}{\partial r} + \frac{1}{r}; \quad \nabla^2 = \mathscr{D}_* \mathscr{D} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2},$$

and

$$\frac{\partial v^0_\phi}{\partial r} - \frac{v^0_\phi}{r} = s; \quad \frac{\partial v^0_\phi}{\partial r} + \frac{v^0_\phi}{r} = 2A.$$

To first order, we get

(i) *Acceleration equations*

$$\rho\left(\frac{\partial u}{\partial t} + \frac{v^0_\phi}{r}\frac{\partial u}{\partial \phi} - \frac{2vv^0_\phi}{r}\right) = -\frac{\partial}{\partial r}(\delta p) + F^{(v)}_r, \tag{A 2}$$

$$F^{(v)}_r = \eta_3\left(\nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \phi}\right) + \frac{\alpha_5 - \alpha_2}{2}\frac{\partial^2 u}{\partial z^2} + \frac{\alpha_5 + \alpha_2}{2}\frac{\partial^2 w}{\partial r\, \partial z} + \frac{\alpha_5 + \alpha_2}{2} s \frac{\partial n_\phi}{\partial z} + \alpha_2\left(\frac{\partial}{\partial t} + \frac{v^0_\phi}{r}\frac{\partial}{\partial \phi}\right)\frac{\partial n_r}{\partial z}, \tag{A 2'}$$

$$\rho\left(\frac{\partial v}{\partial t} + \frac{v^0_\phi}{r}\frac{\partial v}{\partial \phi} + 2Au\right) = -\frac{1}{r}\frac{\partial}{\partial \phi}(\delta p) + F^{(v)}_\phi, \tag{A 3}$$

$$F_\phi^{(v)} = \eta_3\left(\nabla^2 v - \frac{v}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial \phi}\right) + \frac{\alpha_5-\alpha_2}{2}\frac{\partial^2 v}{\partial z^2} + \frac{\alpha_5+\alpha_2}{2}\frac{1}{r}\frac{\partial^2 w}{\partial z\,\partial \phi} + \frac{\alpha_5-\alpha_2}{2}s\frac{\partial n_r}{\partial z} + \alpha_2\left(\frac{\partial}{\partial t} + \frac{v_\phi^0}{r}\frac{\partial}{\partial \phi}\right)\frac{\partial n_\phi}{\partial z}, \tag{A 3$'$}$$

$$\rho\left(\frac{\partial w}{\partial t} + \frac{v_\phi^0}{r}\frac{\partial w}{\partial \phi}\right) = -\frac{\partial}{\partial z}(\delta p) + F_0^{(v)}, \tag{A 4}$$

$$F_z^{(v)} = \eta_1 \nabla^2 w + (\alpha_1+\alpha_5)\frac{\partial^2 w}{\partial z^2} + \frac{\alpha_3+\alpha_6}{2}\mathscr{D}_*(sn_\phi) + \frac{\alpha_6-\alpha_3}{2}s\frac{1}{r}\frac{\partial n_r}{\partial \phi} + \alpha_3\left(\frac{\partial}{\partial t} + \frac{v_\phi^0}{r}\frac{\partial}{\partial \phi}\right)\left(\mathscr{D}_* n_r + \frac{1}{r}\frac{\partial n_\phi}{\partial \phi}\right) + s\frac{1}{r}\frac{\partial n_r}{\partial \phi}, \tag{A 4$'$}$$

with $$\eta_3 = \frac{\alpha_4}{2} \quad \text{and} \quad \eta_1 = \frac{\alpha_4+\alpha_3+\alpha_6}{2}.$$

(ii) *Torque equations*

Elastic part:

$$\Gamma_r^{(e)} = -K_2\frac{\partial}{\partial r}\left(\mathscr{D}_* n_\phi - \frac{1}{r}\frac{\partial n_r}{\partial \phi}\right) - K_1\frac{1}{r}\frac{\partial}{\partial \phi}\left(\mathscr{D}_* n_r + \frac{1}{r}\frac{\partial n_\phi}{\partial \phi}\right) - K_3\frac{\partial^2 n_\phi}{\partial z^2}, \tag{A 5}$$

$$\Gamma_\phi^{(e)} = K_1\frac{\partial}{\partial r}\left(\mathscr{D}_* n_r + \frac{1}{r}\frac{\partial n_\phi}{\partial \phi}\right) - K_2\frac{1}{r}\frac{\partial}{\partial \phi}\left(\mathscr{D}_* n_\phi - \frac{1}{r}\frac{\partial n_r}{\partial \phi}\right) + K_3\frac{\partial^2 n_r}{\partial z^2}. \tag{A 6}$$

Viscous part:

$$\Gamma_r^{(v)} = \gamma_1\left(\frac{\partial n_\phi}{\partial t} + \frac{v_\phi^0}{r}\frac{\partial n_\phi}{\partial \phi}\right) + \alpha_3\frac{1}{r}\frac{\partial w}{\partial \phi} + \alpha_2\frac{\partial v}{\partial z} + \alpha_2 s n_r, \tag{A 5$'$}$$

$$\Gamma_\phi^{(v)} = -\gamma_1\left(\frac{\partial n_r}{\partial t} + \frac{v_\phi^0}{r}\frac{\partial n_r}{\partial \phi}\right) - \alpha_2\frac{\partial u}{\partial z} - \alpha_3\frac{\partial w}{\partial r} - \alpha_3 s n_\phi, \tag{A 6$'$}$$

with $$\gamma_1 = \alpha_3 - \alpha_2.$$

To first order $\Gamma_z^{(e)} = \Gamma_z^{(v)} = 0$, and the torque equations read

$$\Gamma_r^{(t)} = \Gamma_r^{(e)} + \Gamma_r^{(v)} = 0,$$

$$\Gamma_\phi^{(t)} = \Gamma_\phi^{(e)} + \Gamma_\phi^{(v)} = 0.$$

(iii) *Continuity equation*

As usual we have

$$\mathscr{D}_* u + \frac{1}{r}\frac{\partial v}{\partial \phi} + \frac{\partial w}{\partial z} = 0. \tag{A 7}$$

Note 1: *Narrow gap approximation.* In the narrow gap approximation we assume

$$d = r_2 - r_1 \ll r_m = \frac{r_1+r_2}{2}.$$

Then

$$\mathscr{D}_* f = \frac{\partial f}{\partial r} + \frac{f}{r} \sim \frac{\partial f}{\partial r},$$

that is to say that we may neglect terms like f/r of order of f/r_m when compared with $\partial f/\partial r$ of order f/d.

Note 2: Equations (3.2)–(3.7) derive from (A 2)–(A 7) when assuming an axisymmetric fluctuation ($\partial/\partial\phi \equiv 0$) and denoting

$$\eta_2 = \frac{\alpha_4+\alpha_5-\alpha_2}{2}; \quad \eta' = \eta_1+\alpha_5+\alpha_1; \quad \eta'' = \eta_3 - \frac{\alpha_5+\alpha_2}{2},$$

$$\frac{\alpha_6+\alpha_3}{2} = \eta_1-\eta_3 \quad (<0 \quad \text{for MBBA}),$$

$$\frac{\alpha_5-\alpha_2}{2} = \eta_2-\eta_3 \quad (>0),$$

$$\frac{\alpha_5+\alpha_2}{2} = \alpha' \quad (<0 \quad \text{for MBBA}).$$

Appendix B

For fluctuations of the form

$$(n_r, n_\phi) = (N_r, N_\phi)\cos(q_r(r-r_m))\cos q_z z \exp \sigma t,$$

$$(v, w) = (V, W)\cos(q_r(r-r_m))\sin q_z z \exp \sigma t,$$

$$(u) = U \sin(q_r(r-r_m))\cos q_z z \exp \sigma t,$$

$$(\delta p) = P \sin(q_r(r-r_m))\sin q_z z \exp \sigma t,$$

one gets:

Torque equations

$$(\gamma_1\sigma + K_2 q_r^2 + K_3 q_z^2) N_\phi + \alpha_2 q_z V + \alpha_2 s N_r = 0, \tag{B 1}$$

$$(\gamma_1\sigma + K_1 q_r^2 + K_3 q_z^2) N_r - \alpha_2 q_z U - \alpha_3 q_r W + \alpha_3 s N_\phi = 0. \tag{B 2}$$

Force equations

$$2\rho\omega_m V - q_r P - (\eta_2 q_z^2 + \eta'' q_r^2) U - q_z(\alpha' s N_\phi + \alpha_2 \sigma N_r) = 0, \tag{B 3}$$

$$-2\rho A U - (\eta_2 q_z^2 + \eta_3 q_r^2) V - q_z((\eta_2-\eta_3) s N_r + \alpha_2\sigma) N_\phi = 0, \tag{B 4}$$

$$-q_z P - (\eta' q_z^2 + \eta_1 q_r^2) W - q_r((\eta_1-\eta_3) s N_\phi + \alpha_3 \sigma) N_r = 0. \tag{B 5}$$

Continuity equation $$q_r U + q_z W = 0, \tag{B 6}$$

where the inertial contributions of the form $\rho(\partial v_x/\partial t)$ have been neglected in the force equations.

The elimination of U, V, W, P leads to the effective torque equations

$$(\gamma_\phi \sigma + f_\phi - 2As\epsilon) N_\phi + (\alpha_2^* s + 2A\sigma\epsilon') N_r = 0, \tag{B 7}$$

$$(\gamma_r \sigma + f_r - 2\omega_m s\epsilon'') N_r + (\alpha_3^* s - 2\omega_m \sigma\epsilon') N_\phi = 0, \tag{B 8}$$

where $$\gamma_\phi = \gamma_1 - \frac{\alpha_2^2 q_z^2 f}{D}, \quad \gamma_r = \gamma_1 - \frac{g(\alpha_2 q_z^2 - \alpha_3 q_r^2)^2}{D},$$

$$\alpha_2^* = \alpha_2\left(1 - \frac{q_z^2 f(\eta_2-\eta_3)}{D}\right),$$

$$\alpha_3^* = \alpha_3 + g\,\frac{(\alpha_2 q_z^2 - \alpha_3 q_r^2)((\eta_1-\eta_3) q_r^2 - \alpha' q_z^2)}{D}, \tag{B 9}$$

$$f_r = (K_1 q_r^2 + K_3 q_z^2), \tag{B 10}$$

$$f_\phi = K_2 q_r^2 + K_3 q_z^2, \tag{B 11}$$

$$\epsilon = \rho \frac{\alpha_2 q_z^2}{D} (q_r^2(\eta_1 - \eta_3) - q_z^2 \alpha'),$$

$$\epsilon' = \rho \frac{\alpha_2 q_z^2}{D} (\alpha_2 q_z^2 - \alpha_3 q_r^2),$$

$$\epsilon'' = \frac{\rho(\eta_2 - \eta_3) q_z^2(\alpha_2 q_z^2 - \alpha_3 q_r^2)}{D},$$

with

$$\left.\begin{aligned} f &= \eta_1 q_r^4 + (\eta' + \eta'') q_r^2 q_z^2 + \eta_2 q_z^4, \\ g &= \eta_2 q_z^2 + \eta_3 q_r^2, \\ D &= fg + 4A\omega_m \rho^2 q_z^2. \end{aligned}\right\} \tag{B 12}$$

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